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The rate at which a long bubble rises in a vertical tube

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As a viscous fluid in a vertical tube drains under the effect of gravity, a finger of air rises in the tube. The shape of the interface and the rate at which the finger rises is determined numerically for different values of the dimensionless parameter $G = \rho g b^2/T$, where ρ is the density difference between the viscous fluid and the air, b is the radius of the tube, and T is the interfacial tension. A relationship between the Bond number G and the capillary number $Ca = \mu U/T$ is found and compared with the perturbation result of Bretherton (1961), where μ is the viscosity of the fluid and U is the constant velocity at which the finger rises. The numerical results support Bretherton's conclusions for very small values of Ca and extend the relationship between G and Ca to a region where the perturbation expansion is no longer valid. The results are also valid for long bubbles rising in a tube.

1. Introduction

Consider the motion of a long bubble rising steadily in a vertical tube, with small radius b, filled with a viscous fluid of viscosity μ. As discussed in Bretherton (1961) and Batchelor (1967), the shape of the cap and base of a long bubble in a tube are independent of the size of the bubble. Only the midsection of the bubble which is approximately cylindrical with radius βb lengthens as the volume of the bubble is increased. The rate at which the bubble rises also remains unchanged. For this reason, we approximate the rate at which a long bubble rises in a vertical tube by examining the limiting case in which a vertical tube is sealed at one end and filled with a viscous fluid that drains out of the lower end. As the fluid drains, a finger of air rises axisymmetrically up the tube under the effect of gravity g with a constant velocity U (see figure 1). We assume that the viscosity of the fluid inside the finger is small in comparison with μ and can therefore be neglected.

There are three important dimensionless parameters in the problem: the capillary number $Ca = \mu U/T$ which gives the ratio of the viscous force to the force of surface tension, the Bond number $G = \rho g b^2/T$ which is the ratio of the force of gravity to interfacial tension, and β which gives the ratio of the radius of the finger to the radius of the tube. Here, ρ is the density difference between the viscous fluid and the air and T is the interfacial tension. On dimensional grounds, the capillary number Ca and thus the speed U at which the finger rises is a function of G. It is assumed that a fourth parameter, the Reynolds number, is small enough that it can be neglected. Using the numerical methods discussed below, we could easily extend the relationship between Ca and G to include the effect of a non-zero Reynolds number.

Bretherton (1961) determined a relationship between Ca and G for very small values of the capillary number and concluded that the error approached 10% when

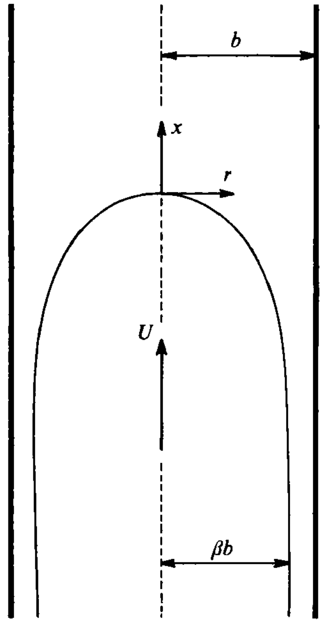

FIGURE 1. Typical bubble profile, $Ca = 5.0 \times 10^{-3}$.

Ca was equal to 8×10^{-5} by estimating the neglected terms in the approximations. The solution was found by examining the shape of the bubble in two different regions. For $Ca \ll 1$, there is an outer region away from the wall of the tube where the viscous terms are not important. The static profile for the cap of the finger is a function of G and can be determined numerically for different values of G. Between the static profile in the outer region and the downstream region where there is a thin film of constant thickness on the wall of the tube, there is a transition region where the viscous terms are important. In this region, the lubrication approximation is used to simplify the equations. The solutions in the two regions must now be matched in an overlap region. Bretherton concluded that only cap profiles with $G > 0.842$ could be matched to a solution in the transition region. For $Ca \ll 1$, he found the following relationship between Ca and G:

$$G - 0.842 \sim 1.25\, Ca^{\frac{2}{9}} + 2.24\, Ca^{\frac{1}{3}}. \tag{1}$$

The bubble only rises when G is greater than the threshold value of 0.842. Singular perturbation methods and matched asymptotic expansions could have been used to formalize Bretherton's approach. This has been done by Park & Homsy (1984) for the case of a two-dimensional finger moving through a horizontal channel in which the force of gravity is neglected.

In this paper, we examine the shape of the finger and its dependence on G for values of the capillary number in the range $1 \times 10^{-4} < Ca < 1 \times 10^{-1}$. As Bretherton mentions in his paper, these values are of greater practical interest than the very small values for which (1) is valid. We compare the results at the lower end of the interval with the results of Bretherton.

2. Formulation of the problem and method of solution

The equations that describe the flow of the viscous fluid as it drains from the tube are the conservation equation,

$$u_x+v_r+\frac{v}{r}=0, \tag{2a}$$

and the Stokes equations,

$$p_x=\mu\left[u_{xx}+u_{rr}+\frac{1}{r}u_r\right]-\rho g, \tag{2b}$$

$$p_r=\mu\left[v_{xx}+v_{rr}+\frac{1}{r}v_r-\frac{v}{r^2}\right]. \tag{2c}$$

We choose a reference frame in which the finger is stationary. In this reference frame, the boundary conditions on the tube $r=b$ are given by

$$u=-U, \quad v=0. \tag{3a, b}$$

In addition, the normal velocity condition and tangential and normal stress conditions must be satisfied on the surface of the finger $x=h(r)$. These conditions can be written

$$\frac{u-h_r v}{(1+h_r^2)^{\frac{1}{2}}}=0, \tag{4a}$$

$$\frac{1-h_r^2}{1+h_r^2}(u_r+v_x)+\frac{2h_r}{1+h_r^2}(u_x-v_r)=0, \tag{4b}$$

$$p-2\mu\left[\frac{u_x-h_r(u_r+v_x)+h_r^2 v_r}{1+h_r^2}\right]=p_0-T\left[\frac{1}{R_1}+\frac{1}{R_2}\right], \tag{4c}$$

where the curvature terms are

$$\frac{1}{R_1}=\frac{-h_{rr}}{(1+h_r^2)^{\frac{3}{2}}}, \quad \frac{1}{R_2}=\frac{-h_r}{r(1+h_r^2)^{\frac{1}{2}}}.$$

As $x\rightarrow-\infty$, the pressure tends to a constant and the velocity of the fluid in the thin film between the finger and the wall, $\beta b\leqslant r\leqslant b$, is given by

$$u\rightarrow-\frac{\rho g}{4\mu}\left[b^2-r^2+2\beta^2b^2\log\left(\frac{r}{b}\right)\right]-U, \tag{5a}$$

$$v\rightarrow 0. \tag{5b}$$

This expression for u is the solution of (2b) which satisfies boundary condition (3a) and interface condition (4b). As $x\rightarrow\infty$, we get

$$u\rightarrow-U, \quad v\rightarrow 0, \tag{5c, d}$$

which is just the velocity of the moving reference frame given in (3a, b). Using (5a, c) and conservation of fluid, we get the following analytical relationship between the three parameters Ca, G, and β which were discussed in the introduction:

$$\frac{Ca}{G}=\frac{1-4\beta^2+3\beta^4-4\beta^4\log\beta}{8\beta^2}. \tag{6}$$

Notice that as the rate of rise of the finger decreases ($Ca\rightarrow 0$), the finger fills the tube ($\beta\rightarrow 1$), but G is left undetermined. Equation (6) is used to eliminate G from the dimensionless forms of (2)–(5). They will now only depend on the parameters β and Ca.

For each value of Ca, we determine the shape of the finger and the value of β by numerically solving (2*a*–*c*) with the appropriate boundary and interface conditions (3)–(5). Knowing β as a function of Ca, we can now determine the relationship between G and Ca.

To solve the free boundary problem for a given value of Ca, we begin with an approximate shape for the interface profile. If the value of Ca is very small, we can use Bretherton's solution; otherwise, we increase the value of Ca in small increments using the interface shape calculated at the previous value. Since the shape of the interface is fixed, it is necessary to drop one of the interface conditions (4*a*–*c*). Usually either the normal velocity condition (4*a*) or the normal-stress boundary condition (4*c*) is dropped. This extra condition is then combined with an iteration method that is repeated until it converges. The rate of convergence of the method or divergence will depend on the choice of the iteration procedure, the mesh size of the grid, and the discretization of the partial differential equations. Silliman & Scriven (1980) determined the interface shape for the two-dimensional discharge of liquid from a sharp-edged slot using the two different approaches. They found, using a different iteration method to the one discussed below, that for small values of Ca the convergence is faster if the normal-stress boundary condition is dropped and for large values of Ca the convergence is faster if the normal velocity condition is used to determine the shape. Since we are primarily interested in small values of Ca, we drop the normal-stress condition.

The fixed domain problem is solved using finite difference methods on a composite mesh composed of a curvilinear grid which follows the curved interface, and a rectilinear grid which is parallel to the wall of the tube. A typical curvilinear grid has 55 gridpoints along the interface and 7 gridpoints perpendicular to the interface. A typical rectilinear grid has 40 points in the x-direction and 30 points in the y-direction. Many of the rectilinear gridpoints are located in the interior of the finger and are not used in calculating the solution. The curvilinear grid is used to get accurate results for the fluid flow near the interface which are needed for the iteration procedure. The two grids overlap in the interior of the fixed domain. Interpolation equations are used to connect the solution on the two overlapping grids. The use of overlapping grids can be easily applied to problems with very complex domains. This technique is also flexible enough to allow for stretched grids so that the number of grid points is greatest in regions where they are needed most. For complete details of this procedure, see Reinelt & Saffman (1985) or Kreiss (1983).

Once the solution is determined on the fixed domain, we substitute the results into the normal-stress boundary condition (4*c*). The shape of the finger is now changed to satisfy this condition to within a given tolerance. In changing the shape of the free boundary, it is efficient to express the shape of the interface using as few parameters as possible. This speeds up the iteration process discussed below. The following function of $\bar{r}$ is used to describe the shape of the interface.

$$\bar{x} = \frac{1}{k} \log\left\{\left[1-\left(\frac{\bar{r}}{\beta}\right)^2\right]\left[1+\left(\frac{\bar{r}}{\beta}\right)^2 \sum_{j=0}^{m} c_j T_{2j}\left(\frac{\bar{r}}{\beta}\right)\right]\right\}, \tag{7}$$

where $0 \leqslant \bar{r} < \beta$. Here, $\bar{x}$ and $\bar{r}$ are the original variables scaled by the tube radius b. The parameters β, c_0, $c_1, \ldots, c_m$ determine the shape of the finger. The log term and the first term in brackets takes into account the asymptotic behaviour of the shape of the interface as $\bar{x} \rightarrow -\infty$

$$\bar{r} \sim \beta - A \exp(k\bar{x}). \tag{8}$$

The decay rate k in (7) and (8) is determined by an analytical expansion of the solution as $\bar{x} \rightarrow -\infty$. This expansion is described in the next section. The functions T_{2j} are the even Chebyshev polynomials. They are chosen because it is expected that the series in (7) will converge rapidly given the distribution of gridpoints along the finger. By this we mean that there are many more gridpoints near β, corresponding to the gridpoints on the sides of the finger, than near the tip of the finger at $\bar{r} = 0$. This is characteristic of the Chebyshev abscissae. The above expansion works well since we only need eight parameters ($m = 6$). The problem is now reduced to finding the values of these eight parameters that best satisfy the normal-stress condition.

The problem was also solved at a number of different Ca values using eleven parameters ($m = 9$). These additional three parameters were all $O(10^{-5})$ or smaller. Including these terms did not change the first five significant digits in the value of the finger width β used to determine the relationship between the capillary number Ca and the Bond number G. The change in shape of the interface profile was also very small.

To determine the values of the parameters that best solve the normal-stress condition, we perturb each of their initial values independently. This requires constructing a new curvilinear grid and solving a new set of equations on a fixed grid for each perturbation. The computing time needed to construct the new grids is a small percentage of the time used in the overall problem. The time needed to solve the new equations can be greatly reduced by observing that the new equations are just perturbations of the original equations. By using the LU decomposition of the original equations and their solution, we can solve these new sets of equations using a couple of forward and backward substitutions, see Reinelt & Saffman (1985).

The above solutions are now used to calculate the Jacobian of the normal-stress boundary condition (4c) with respect to the parameters β, $c_0, c_1, \ldots, c_m$. This leads to a set of linear equations equivalent to the equations one would get using Newton's method except that the number of equations, equal to the number of gridpoints on the interface, is much greater than the number of parameters; thus, these equations are solved in a least-squares sense. This means that the parameters are chosen such that the error in solving the normal-stress boundary condition at each gridpoint on the interface is minimized with respect to the Euclidean norm. For each value of Ca, the entire procedure requires only three or four iterations before the correct interface shape is found.

3. Determination of the decay rate k

The expansion of the interface given in (7) involves the decay rate k which is determined analytically in terms of β, Ca, and G by examining the asymptotic behaviour of the solution as $x \rightarrow -\infty$. To accomplish this, we introduce the stream function $\psi(x, r)$ which is related to the velocity components by

$$u = \frac{1}{r}\psi_r, \quad v = -\frac{1}{r}\psi_x. \tag{9a, b}$$

This allows us to rewrite (2a–c) in terms of a single fourth-order equation for ψ

$$E^4\psi = 0, \tag{10}$$

where
$$E^2\psi = \psi_{rr} - \frac{1}{r}\psi_r + \psi_{xx}.$$

As $x \to -\infty$, the asymptotic shape of the finger profile given in (8) is

$$r \sim \beta b - bA \exp\left(\frac{kx}{b}\right), \tag{11}$$

and the stream function ψ has the asymptotic expansion

$$\psi(x, r) \sim S_0(r) + S_1(r) \exp\left(\frac{kx}{b}\right). \tag{12}$$

The function $S_0(r)$ is determined by integrating (5a) using (9a)

$$S_0(r) = -\frac{\rho g}{4\mu}\left[\tfrac{1}{2}b^2r^2 - \tfrac{1}{4}r^4 + \beta^2 b^2\left\{r^2 \log\left(\frac{r}{b}\right) - \tfrac{1}{2}r^2\right\}\right] - \tfrac{1}{2}Ur^2 + \text{constant}. \tag{13}$$

The constant is specified by setting ψ equal to zero on the interface (i.e. $S_0(\beta b) = 0$).

To get an equation for $S_1(r)$, we substitute the expansion for ψ given in (12) into (10). The solution involves Bessel functions and is given by

$$S_1(r) = Br^2 J_0\left(\frac{kr}{b}\right) + CrJ_1\left(\frac{kr}{b}\right) + Dr^2 Y_0\left(\frac{kr}{b}\right) + ErY_1\left(\frac{kr}{b}\right). \tag{14}$$

The five arbitrary constants, A, B, C, D, and E in (11) and (14) are determined by satisfying the two boundary conditions on the wall of the tube and the three interface conditions. These five equations will only have a solution, other than the trivial solution (all constants equal to zero), if the determinant of the matrix given below is equal to zero:

$$\begin{bmatrix} a_1 & J_1(k\beta) & 0 & Y_1(k\beta) & 0 \\ a_2 & J_0(k\beta) & J_1(k\beta) & Y_0(k\beta) & Y_1(k\beta) \\ a_3 & 0 & J_1(k\beta) - k\beta J_0(k\beta) & 0 & Y_1(k\beta) - k\beta Y_0(k\beta) \\ 0 & J_0(k) & \beta J_1(k) & Y_0(k) & \beta Y_1(k) \\ 0 & 2J_0(k) - kJ_1(k) & \beta k J_0(k) & 2Y_0(k) - kY_1(k) & \beta k Y_0(k) \end{bmatrix}, \tag{15}$$

where

$$a_1 = 1 + \frac{G}{4Ca}(1 - \beta^2 + 2\beta^2 \log \beta) - \frac{G}{2k^2 Ca},$$

$$a_2 = -\frac{1}{k\beta}\left[1 + \frac{G}{4Ca}(1 - \beta^2 + 2\beta^2 \log \beta)\right],$$

$$a_3 = \frac{1}{2Ca}\left(1 + \frac{1}{k^2\beta^2}\right) - k\beta\left[1 + \frac{G}{4Ca}(1 - \beta^2 + 2\beta^2 \log \beta) - \frac{G}{2k^2 Ca}\right].$$

This gives us a relationship between k, β, Ca, and G. The G dependence in the relationship between the four parameters is eliminated by using (6); thus, the decay rate k given in (7) will depend only on β and Ca.

The matrix given above is similar to the one given by Cox (1962) who performed experiments on driving a viscous fluid out of a horizontal tube. The only difference occurs in the entries a_1, a_2, and a_3 which now include the effect of gravity. A more detailed derivation of the above matrix is given by Cox. In addition, as $Ca \to 0$ which corresponds to $\beta \to 1$ from (6), we get the asymptotic result

$$k \sim \left(\frac{3G}{1-\beta}\right)^{\frac{1}{3}}, \tag{16}$$

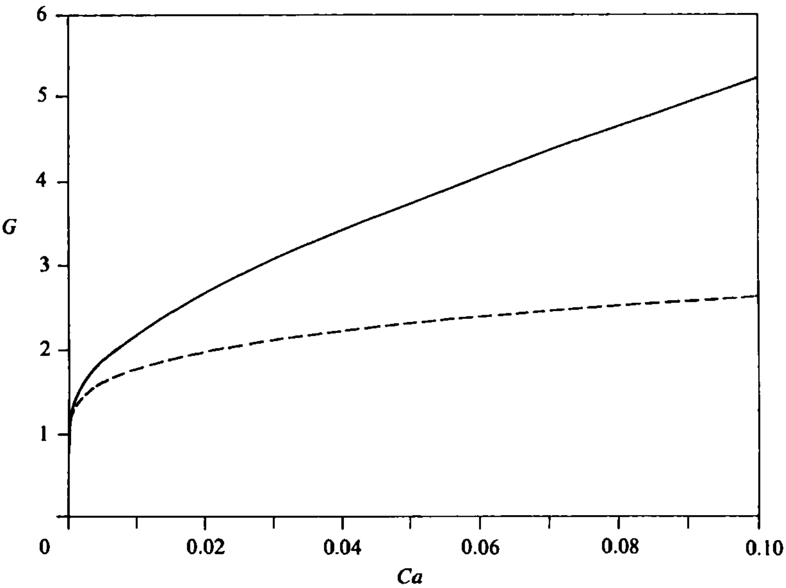

FIGURE 2. The Bond number $G = \rho g b^2/T$ as a function of the capillary number $Ca = \mu U/T$. ——, numerical result; ---, Bretherton's perturbation result.

after very tedious algebra. This result checks with the perturbation result of Bretherton.

4. Results

The problem is solved for values of the capillary number in the range $1 \times 10^{-4} < Ca < 1 \times 10^{-1}$. In §2, we discussed the numerical solution of the differential equations which gives β in terms of Ca and through (6) gives the Bond number G as a function of the capillary number Ca. Plots of G and β versus Ca are given in figures 2 and 3 respectively. The broken line is a plot of G versus Ca using the perturbation result of Bretherton. Notice that the numerical and perturbation results are very different except when Ca is nearly zero. This was expected because the perturbation result is only valid for very small values of Ca.

To make a better comparison between the two results as $Ca \rightarrow 0$, we plot G versus Ca using a logarithmic scale for the capillary number (see figure 4). The actual values of Ca for which G and the finger profile were determined are also shown. In this plot, it seems clear that the numerical result approaches Bretherton's result asymptotically as $Ca \rightarrow 0$. It should be noted that even when $Ca = 1 \times 10^{-4}$ the difference between G and the value of 0.842 at which the finger will no longer rise is still on the order of 10^{-1}. This is because the perturbation expansion contains fractional powers of Ca; thus, extremely small values of Ca are required to make the right-hand side of (1) small. This also means that the finger rises very slowly until G reaches about 1.2. As mentioned in the introduction, these results for the draining problem are also a valid approximation for long bubbles rising in vertical tubes.

The mathematical methods discussed above worked well in the treatment of this problem. The composite mesh is able to provide an accurate solution near the interface so that the correct shape of the finger can be determined. This method also

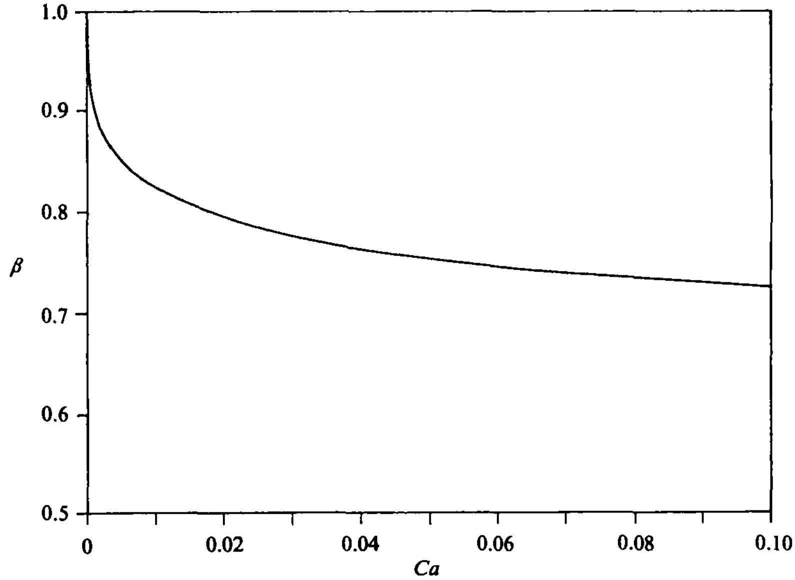

FIGURE 3. The finger width β as a function of the capillary number Ca.

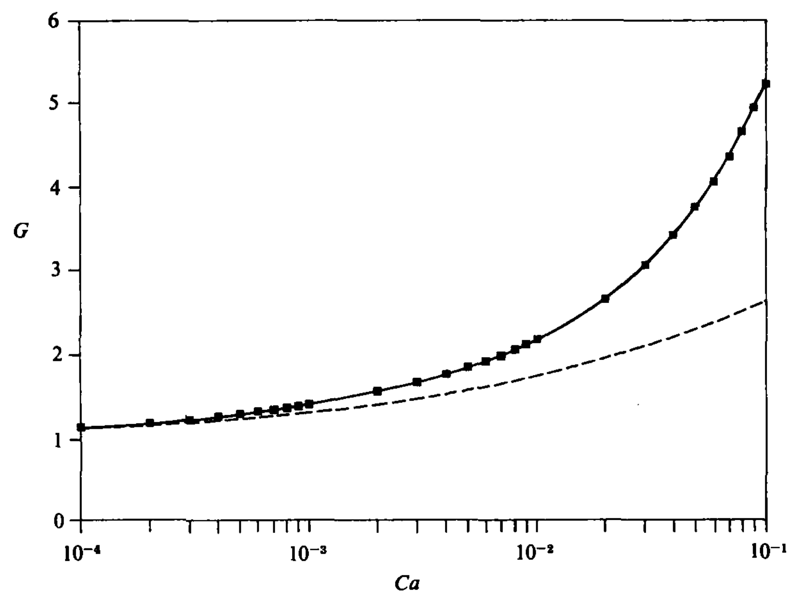

FIGURE 4. G *vs.* Ca (Ca plotted on a logarithmic scale). —■—, numerical result; ---, perturbation result.

has the advantage that it can include the convective terms without much difficulty unlike boundary integral methods which are often used to solve free surface creeping flow problems, see for example Youngren & Acrivos (1976). This would allow us to find the dependence of the shape of the finger and the rate at which it rises on both G and the Reynolds number.

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